

Discrete-time trawl processes with long memory

P. Doukhan*, A. Jakubowski†, S.R.C. Lopes‡ and D. Surgailis§

October 18, 2016

Abstract

We introduce a class of discrete time stationary trawl processes taking real or integer values and written as sums of past values of independent ‘seed’ processes on shrinking intervals (‘trawl heights’). Related trawl processes in continuous time were studied in Barndorff-Nielsen (2011) and Barndorff-Nielsen et al. (2014), however in our case the i.i.d. seed processes can be very general and need not be infinitely divisible. In the case when the trawl height decays with the lag as $j^{-\alpha}$ for some $1 < \alpha < 2$, the trawl process exhibits long memory and its covariance decays as $j^{1-\alpha}$. We show that under general conditions on generic seed process, the normalized partial sums of such trawl process may tend either to a fractional Brownian motion or to an α -stable Lévy process.

Keywords: trawl process, integer and continuous-valued time series, long memory, fractional Brownian motion, Lévy process.

AMS Classification subjects 2010

60G22 Fractional processes, including fractional Brownian motion

60G51 Processes with independent increments; Lévy processes

60G99 Trawl process

60K99 Long range memory process

1 Introduction

The present paper introduces a class of stationary random processes of the form

$$X_k = \sum_{j=0}^{\infty} \gamma_{k-j}(a_j), \quad k \in \mathbb{Z} \quad (1.1)$$

where $\gamma_k = \{\gamma_k(u), u \in \mathbb{R}\}$ are i.i.d. copies of a generic process $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ tending to zero in probability as $u \rightarrow 0$, and $a_j \in \mathbb{R}$ for $j \in \mathbb{N}$, $\lim_{j \rightarrow \infty} a_j = 0$ are deterministic

*UMR AGM8088, University Cergy-Pontoise.

†Nicolaus Copernicus University, Torun.

‡Federal University of Rio Grande de Sul, UFRGS.

§Vilnius University.

numbers. Clearly, (1.1) includes the class of causal moving averages $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$ in i.i.d. r.v.s $\{\xi, \xi_k\}$, which correspond to a trivial process $\gamma = \{\gamma(u) = \xi u, u \in \mathbb{R}\}$. In as follows, we call $X = \{X_k, k \in \mathbb{Z}\}$ the *trawl process* corresponding to the *seed process* $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ and *trawl* $a = \{a_j, j \geq 0\}$. The above terminology is borrowed from Barndorff-Nielsen et al. (2014) [4] which considered a related class of trawl processes in continuous time represented as stochastic integrals

$$Y_t = \int_{(-\infty, t] \times \mathbb{R}} \mathbf{1}(x \in (0, d_{t-s})) L(dx, ds), \quad t \in \mathbb{R} \quad (1.2)$$

where $L(dx, ds)$ is a homogeneous Lévy measure on \mathbb{R}^2 , with independent values on disjoint sets, and $\{d_t, t \in \mathbb{R}_+\}$ is a deterministic function satisfying certain conditions. In the case when this function takes constant values $d_t = a_j$, if $t \in (j, j+1]$, for $j = 0, 1, \dots$, the discretized process $\{Y_k, k \in \mathbb{Z}\}$ in (1.2) coincides with $\{X_k, k \in \mathbb{Z}\}$ in (1.1) with independent increment (Lévy) seed process $\left\{ \zeta(u) = \int_{(0, u] \times (0, 1]} L(dx, ds), u \in \mathbb{R} \right\}$. Clearly, an integer-valued seed process $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ in (1.1) results in an integer-valued trawl process $\{X_k, k \in \mathbb{Z}\}$, similarly as in the case of continuous-time trawl processes of (1.2) studied in [4]. On the other hand, the discrete-time set-up allows us to consider very general seed processes γ which need not be infinitely divisible or have independent increments as in [4]. ([3], page 22) note that trawl processes represent a flexible class of stochastic processes which can be used to model serially dependent count data and other stationary time series, where the marginal distribution and the autocorrelation structure can be modeled independently from each other. Particularly, trawl processes can exhibit long memory or long-range dependence, which is usually associated with the divergence of the covariance series: $\sum_{k \in \mathbb{Z}} |\text{Cov}(X_0, X_k)| = \infty$, see [12], and which occurs in models (1.1) and (1.2) when the trawl function decays sufficiently slowly with the lag, see [4] and § 2 below. ([4], figure 6) exhibit sample paths and autocorrelation graphs of integer-valued trawl process with long-memory trawl function showing a remarkably slow decay and a disagreement between true and sample autocorrelations based on a very large sample length.

The main question studied in this paper, which is also one of the basic questions for statistical applications of trawl processes, is the rate of convergence and the limit distribution of the sample mean. We prove that for trawl process with long-memory trawl function a_j decaying as $j^{-\alpha}$, $1 < \alpha < 2$ this limit distribution is either α -stable or Gaussian, moreover, a non-Gaussian stable limit is typical for integer valued seed (and trawl) process, while a Gaussian limit occurs for ‘continuous’ seed processes, e.g. diffusions or stochastic volatility processes. We note that our non-Gaussian result contradicts the conjecture in ([4], page 708) about a Gaussian partial sums limit for long-memory trawl process in (1.2). In particular, for a standard Poisson seed process γ and $a_j \sim c_0 j^{-\alpha}$, $1 < \alpha < 2$ we obtain, with

$H = (3 - \alpha)/2$, a sequence of processes

$$Z_n(t) = \frac{1}{n^H} \sum_{j=1}^{[nt]} (X_j - \mathbb{E}X_j)$$

whose second order moments converge to those of a fractional Brownian motion, B_H with index H :

$$\lim_{n \rightarrow \infty} \text{Cov}(Z_n(s), Z_n(t)) = \text{Cov}(B_H(s), B_H(t)), \quad \forall s, t.$$

Moreover, $Z_n(t) \rightarrow 0$ in probability (the process is evanescent) but $n^{H-\frac{1}{\alpha}} Z_n(t)$ converges to a non-trivial limit which is an α -stable Lévy process. (Note $H - \frac{1}{\alpha} = \frac{(2-\alpha)(\alpha-1)}{2\alpha} > 0$ since $1 < \alpha < 2$.)

A similar phenomenon (convergence of the partial sums process to a Lévy stable process) occurs for a number of long-range dependent stationary processes with finite variance, see [28], [29], [17], or [23], [30], [19], [27], [16], [24] and the references therein, although in most of the literature this convergence is limited to finite-dimensional distributions. For $M/G/\infty$ queue with heavy-tailed activity periods, the adequate functional convergence was proved in [25]. Since the limiting stable processes in these works have independent increments, the above behavior is sometimes called ‘distributional short-range dependence’ in contrast to ‘distributional long-range dependence’ occurring when the limit of the partial sums process has dependent increments. See [8], [20]. See also [21] for a nice discussion of stable and Gaussian limits under long-range dependence.

2 Discrete-time trawl process

2.1 Existence of discrete-time trawl process

Let $\gamma_k = \{\gamma_k(u), u \in \mathbb{R}\}$ be i.i.d. copies of a generic *seed process* $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ with finite variance $g(u) = \text{Var}(\gamma(u))$ and mean $\mu(u) = \mathbb{E}\gamma(u)$ tending to zero as $u \rightarrow 0$ so that $\gamma(0) = 0$ and $\gamma(u) \rightarrow_{\mathbb{P}} 0$ as $u \rightarrow 0$. A trawl $a = \{a_j \geq 0, j \in \mathbb{N}\}$ is a deterministic sequence such that $\lim_{j \rightarrow \infty} a_j = 0$. We shall assume that

$$|\mu(u)| = \mathcal{O}(g(u)) \rightarrow 0 \quad (u \rightarrow 0) \tag{2.3}$$

and

$$\sum_{j=0}^{\infty} g(a_j) < \infty. \tag{2.4}$$

The trawl process $X = \{X_k, k \in \mathbb{Z}\}$ corresponding to trawl $a = \{a_j \geq 0, j \in \mathbb{N}\}$ and seed process $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ is defined as

$$X_k = \sum_{j=0}^{\infty} \gamma_{k-j}(a_j), \quad k \in \mathbb{Z}. \tag{2.5}$$

Let

$$\rho(u, v) = \text{Cov}(\gamma(u), \gamma(v)), \quad (u, v \in \mathbb{R}) \quad (2.6)$$

denote the covariance function of the seed process γ . The following statement is an easy consequence of the Kolmogorov three series theorem.

Proposition 1. *Let conditions (2.3) and (2.4) be satisfied. Then the series in (2.5) converges a.s. and in mean square for any $k \in \mathbb{Z}$, and defines a stationary process with mean $\mathbb{E}X_k = \sum_{j=0}^{\infty} \mu(a_j)$ and covariance function*

$$\text{Cov}(X_0, X_k) = \sum_{j=0}^{\infty} \rho(a_j, a_{j+k}), \quad k \in \mathbb{N}. \quad (2.7)$$

Clearly, if the seed process takes integer values: $\gamma(u) \in \mathbb{Z}$, $u \in \mathbb{R}$, this property also holds for the trawl process: $X_k \in \mathbb{Z}$ ($\forall k \in \mathbb{Z}$). The following examples show that the class of trawl processes is very large.

Example 1 (Random line seed process). Let $\gamma(u) = \xi u$, $u \in \mathbb{R}$, where ξ is a r.v. with zero mean and variance $\sigma^2 < \infty$. Then $\mu(u) = 0$, $g(u) = \sigma^2 u^2$ and condition (2.4) translates to $\sum_{j=0}^{\infty} a_j^2 < \infty$. Then X in (2.5) is a moving-average:

$$X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, \quad (2.8)$$

where $\{\xi_k, k \in \mathbb{Z}\}$ are i.i.d. copies of ξ .

Example 2 (Brownian motion seed process). Let $\gamma(u) = B(u)$, $u \in \mathbb{R}_+$, where B is a Brownian motion with zero mean and covariance $\mathbb{E}B(u)B(v) = u \wedge v$ and $a_j \geq 0$. Then X in (2.5) is a stationary Gaussian process with zero mean and covariance $\text{Cov}(X_0, X_k) = \sum_{j=0}^{\infty} a_j \wedge a_{k+j}$, $k \in \mathbb{N}$. Particularly, if $a_j = a^j$, $a \in (0, 1)$ then $\text{Cov}(X_0, X_k) = a^k / (1 - a)$ and X in (2.5) agrees with an AR(1) process written as a moving-average in (2.8) with Gaussian innovations $\xi_k \sim \mathcal{N}(0, \sigma^2)$ and $\sigma^2 = 1 + a$.

Example 3 (Poisson and Bernoulli seed processes). Let $\gamma(u) = P(u)$, $u \in \mathbb{R}_+$, where P is a Poisson process with mean $\mu(u) = u$, covariance $\rho(u, v) = \text{Cov}(P(u), P(v)) = u \wedge v$ and $a_j \geq 0$. Then X in (2.5) is a stationary process with mean $\mathbb{E}X_k = \sum_{j=0}^{\infty} a_j$ and the same covariance as in Example 2. Moreover, X_k takes integer values and has a Poisson marginal distribution with mean $\mathbb{E}X_0$.

The above example can be generalized by considering a mixed Poisson seed process $\gamma(u) = P(u\zeta)$, where P is as above and $\zeta > 0$ is a random variable with $\mathbb{E}\zeta < \infty$, independent of P . Particularly, [6] proved that when ζ is exponentially distributed then $P(u\zeta)$ has negative binomial marginal distribution.

The Bernoulli seed process is defined by $\gamma(u) = \mathbb{1}(U \leq u)$, where $U \sim \mathcal{U}[0, 1]$ is a uniformly distributed random variable. Note also

$$\begin{aligned} \mu(u) &= u\mathbb{E}\zeta, & \rho(u, v) &= (u \wedge v)\mathbb{E}\zeta + uv\text{Var}(\zeta) & (\gamma \text{ is a mixed Poisson process}), \\ \mu(u) &= u, & \rho(u, v) &= u \wedge v - uv & (\gamma \text{ is a Bernoulli process}). \end{aligned}$$

Further examples of trawl processes can be found in § 3.1 (Examples 4-5) and § 3.2 (Example 6). As explained in § 1, this paper is focused on long memory properties and the behavior of the partial sums process of stationary trawl process X in (2.5).

2.2 Second order properties of discrete-time trawl process

The covariance function $\text{Cov}(X_0, X_k)$ in (2.7) depends both on the trawl $a = \{a_j\}$ and on the covariance function $\rho(u, v)$ of the seed process. In order to characterize long memory property in terms of the trawl $a = \{a_j\}$ alone, it is convenient to impose a linear growth condition on the variance $g(u) = \text{Var}(\gamma(u))$ at the origin $u = 0$:

$$g(u) = |u|(1 + o(1)), \quad u \rightarrow 0. \quad (2.9)$$

Under (2.9), condition (2.4) is equivalent to the summability of the trawl sequence:

$$\sum_{j=0}^{\infty} |a_j| < \infty. \quad (2.10)$$

Moreover, for obtaining more precise decay of the covariance function in (2.7) we also assume that

$$\rho(u, v) = (|u| \wedge |v|)(1 + o(1)), \quad \text{as } u, v \rightarrow 0, \quad uv > 0. \quad (2.11)$$

Clearly, the trawl processes in Examples 2 and 3 satisfy (2.9) and (2.11) provided the seed processes in these examples are suitably extended to negative $u < 0$. Denote by $S_n = \sum_{k=1}^n X_k$ the partial sums process of the trawl process in (2.5).

Proposition 2. (i) Assume conditions (2.3), (2.9), (2.11) and

$$a_j = c_0 j^{-\alpha}(1 + o(1)), \quad j \rightarrow \infty \quad (\exists c_0 \neq 0, 1 < \alpha < 2). \quad (2.12)$$

Then

$$\text{Cov}(X_0, X_k) = c_1 k^{1-\alpha}(1 + o(1)), \quad k \rightarrow \infty \quad (2.13)$$

and

$$\text{Var}(S_n) = \sum_{k,l=1}^n \text{Cov}(X_k, X_l) \sim c_2 n^{3-\alpha} \gg n, \quad n \rightarrow \infty, \quad (2.14)$$

where $c_1 = c_0/(\alpha - 1)$, and $c_2 = 2c_1/(2 - \alpha)(3 - \alpha)$.

(ii) Assume conditions (2.3), (2.9),

$$|\rho(u, v)| \leq C(|u| \wedge |v|) \quad (u, v \in \mathbb{R}) \quad (2.15)$$

and

$$\sum_{j=1}^{\infty} j|a_j| < \infty. \quad (2.16)$$

Then

$$\sum_{k=1}^{\infty} |\text{Cov}(X_0, X_k)| < \infty \quad (2.17)$$

and

$$\text{Var}(S_n) = n \sum_{|k| < n} \left(1 - \left|\frac{k}{n}\right|\right) \text{Cov}(X_k, X_0) \sim \sigma^2 n, \quad (2.18)$$

where $\sigma^2 = \sum_{k \in \mathbb{Z}} \text{Cov}(X_0, X_k)$.

Remark 1. The estimation of the parameter of interest needs additional work: it will be considered in further papers.

Proof. (i) Let $c_0 > 0$ in (2.12), the case $c_0 < 0$ follows analogously. Then $a_j > 0$, and $a_{k+j} > 0$ hold for all $k \geq 1$ and $j > j_0$, where j_0 is large enough. Moreover, for any $\epsilon > 0$ there exists $j_0 < j_\epsilon < \infty$ such that

$$a_{j+k} < a_j, \quad \text{for all } \forall j_\epsilon < j < k/2\epsilon, \quad \forall k \geq 2\epsilon j_\epsilon. \quad (2.19)$$

Indeed, by (2.12) we have that for any $\epsilon > 0$ there exists $j_\epsilon > j_0 > 0$ such that $a_j > c_0 j^{-\alpha}(1 - \epsilon)$, $a_{k+j} < c_0(j + k)^{-\alpha}(1 + \epsilon)$ and therefore

$$\left(\frac{a_{j+k}}{a_j}\right)^{\frac{1}{\alpha}} < \frac{j}{j+k} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\frac{1}{\alpha}}, \quad \forall j > j_\epsilon, \quad \forall k \geq 1.$$

Since $((1 + \epsilon)/(1 - \epsilon))^{\frac{1}{\alpha}} < 1 + 2\epsilon$ if $\epsilon > 0$ is small enough, relation (2.19) follows since $j/(j + k) \leq 1/(1 + 2\epsilon)$ for $1 \leq j < k/2\epsilon$. Next, for sufficiently large k ($k > 2\epsilon j_\epsilon$) split $k^{\alpha-1} \text{Cov}(X_0, X_k) = \sum_{j=0}^{\infty} k^{\alpha-1} \rho(a_j, a_{k+j}) = \sum_{i=1}^3 I_{i,k}$, where

$$I_{1,k} = \sum_{0 \leq j \leq j_\epsilon} \dots, \quad I_{2,k} = \sum_{j_\epsilon < j < k/2\epsilon} \dots, \quad I_{3,k} = \sum_{j \geq k/2\epsilon} \dots$$

By (2.9), (2.12) and Cauchy-Schwartz inequality, for any fixed $\epsilon > 0$ and $1 \leq j \leq j_\epsilon$,

$$|\rho(a_j, a_{k+j})| \leq g(a_j)^{\frac{1}{2}} g(a_{k+j})^{\frac{1}{2}} \leq C|a_{k+j}|^{\frac{1}{2}} \leq Ck^{-\frac{\alpha}{2}}, \quad k \rightarrow \infty$$

implying

$$|I_{1,k}| \leq Ck^{\alpha-1} k^{-\frac{\alpha}{2}} = O(k^{-(1-\frac{\alpha}{2})}) = o(1), \quad k \rightarrow \infty.$$

Next, by (2.11) and (2.12), $|\rho(a_j, a_{k+j})| \leq C|a_j| \wedge |a_{k+j}| \leq Cj^{-\alpha}$, ($\forall j, k \geq 1$) and therefore

$$I_{3,k} \leq Ck^{\alpha-1} \sum_{j \geq k/2\epsilon} j^{-\alpha} \leq C\epsilon^{\alpha-1}$$

can be made arbitrarily small uniformly in $k \geq 1$ by choosing $\epsilon > 0$ small enough. Finally, by (2.19) and (2.11),

$$I_{2,k} = c_0 k^{\alpha-1} \sum_{j_\epsilon < j < k/2\epsilon} \frac{1 + \delta_{j,k}}{(k+j)^\alpha}, \quad (2.20)$$

where $\sup_{j \geq 1} |\delta_{j,k}| = 0$ as $k \rightarrow \infty$. Note that for each $\epsilon > 0$, as $k \rightarrow \infty$

$$\begin{aligned} J_k(\epsilon) &:= k^{\alpha-1} \sum_{j_\epsilon < j < k/2\epsilon} (k+j)^{-\alpha} = \frac{1}{k} \sum_{\frac{j_\epsilon}{k} < \frac{j}{k} < 1/2\epsilon} \frac{1}{\left(1 + \frac{j}{k}\right)^\alpha} \\ &\rightarrow \int_0^{1/2\epsilon} \frac{dx}{(1+x)^\alpha} = \frac{1}{\alpha-1} (1 - (2\epsilon)^{\alpha-1}). \end{aligned} \quad (2.21)$$

According to (2.20) and (2.21), for any $\delta > 0$ and any $\epsilon_0 > 0$ one can find $0 < \epsilon < \epsilon_0$ and $K_0 > 0$ such that $|I_{2,k} - c_0/(\alpha-1)| < \delta$ holds for all $k > K_0$. This proves (2.13) while (2.14) follows from (2.13), see e.g. ([12], proposition 3.3.1).

(ii) It suffices to prove (2.17) since (2.18) follows from (2.17) and the dominated convergence theorem. According to (2.7), (2.15), (2.16),

$$\begin{aligned} \sum_{k=1}^{\infty} |\text{Cov}(X_0, X_k)| &\leq C \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} |a_j| \wedge |a_{j+k}| \\ &\leq C \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} |a_{j+k}| \leq C \sum_{k=1}^{\infty} k |a_k| < \infty. \end{aligned}$$

Proposition 2 is proved. \square

3 Partial sums limits of trawl processes

We shall consider two typical cases of the seed process γ in (1.1):

Case 1: $\gamma(u), u \geq 0$ is centered: $\mu(u) = 0$ and a.s. continuous (e.g., a Brownian motion).

Case 2: $\gamma(u), u \geq 0$ is a pure jump process (a typical example is a Poisson process with $\mu(u) = g(u) = u$).

Particularly, in Examples 2 and 3 of γ (Brownian motion and Poisson process) and a regularly decaying trawl $a = \{a_j\}$ in (2.16) with exponent $1 < \alpha < 2$ the conditions of Proposition 2 (i) are satisfied and the covariance function of the trawl process decays as $k^{1-\alpha}$, see (2.13). The last fact implies that the variance of $S_n = \sum_{k=1}^n X_k$ grows faster than n , see (2.18).

In the following subsections we detail conditions on the seed process $\{\gamma(u), u \in \mathbb{R}\}$ which guarantee that the partial sums process of the trawl process $\{X_k\}$ with regularly decaying trawl (2.12) tends to either a Gaussian process (fractional Brownian motion with Hurst parameter $H = (3 - \alpha)/2 \in (1/2, 1)$ (Case 1) or to a α -stable Lévy process (Case 2).

The following decomposition of the partial sums process as a sum of independent random variables is crucial for the proofs of Theorem 1 and Theorem 2.

Lemma 1 (Decomposition). *We have*

$$S_n = \sum_{k=1}^n X_k = \sum_{s=-\infty}^n Z_{s,n}, \quad \text{where} \quad Z_{s,n} = \sum_{k=1 \vee s}^n \gamma_s(a_{k-s}). \quad (3.22)$$

Then the random variables $(Z_{s,n})_{s \leq n}$ are independent.

Write $\rightarrow_{f.d.d.}$ for the weak convergence of finite-dimensional distributions and $\rightarrow_{\mathcal{D}(J_1)}$ and $\rightarrow_{\mathcal{D}(M_1)}$ for the weak convergence of random elements in the Skorohod space $D[0, 1]$ endowed with the J_1 -topology and the M_1 -topology, respectively. For the definition of these topologies, see Skorohod [26] or [5], [22], [25]. Denote $|\mu|_{2+\delta}(u) = \mathbb{E}|\gamma(u)|^{2+\delta}$ the absolute $(2 + \delta)$ -moment of the seed process.

3.1 Gaussian scenario (Case 1)

Theorem 1.

(i) Assume $\mu(u) = \mathbb{E}\gamma(u) = 0$, (2.9), (2.11), (2.12) and

$$|\mu|_{2+\delta}(u) = \mathcal{O}(|u|^{\frac{2+\delta}{2}}), \quad (u \rightarrow 0, \exists \delta > 0). \quad (3.23)$$

Then

$$\frac{1}{n^H} S_{[nt]} \rightarrow_{\mathcal{D}(J_1)} \sqrt{c_2} B_H(t), \quad H = \frac{3-\alpha}{2} \quad (3.24)$$

where B_H is fractional Brownian motion with variance $\mathbb{E}B_H^2(t) = t^{2H}$ and c_2 is defined in (2.14).

(ii) Assume $\mu(u) = \mathbb{E}\gamma(u) = 0$, (2.15), (2.16), (3.23) and $\sigma^2 = \sum_{k \in \mathbb{Z}} \text{Cov}(X_0, X_k) \neq 0$.

Then

$$\frac{1}{\sqrt{n}} S_{[nt]} \rightarrow_{f.d.d.} \sigma B(t), \quad (3.25)$$

where B is a Brownian motion with variance $\mathbb{E}B^2(t) = t$.

In addition, if $\sum_{k=1}^{\infty} \sqrt{|a_k|} < \infty$, then the finite dimensional convergence in (3.25) can be replaced by $\rightarrow_{\mathcal{D}(J_1)}$.

(iii) Assume the same conditions as in (ii) except that (3.23) is replaced by

$$|\mu|_{2+\delta}(u) = \mathcal{O}(u) \quad (u \rightarrow 0) \quad \text{and} \quad \sum_{j=0}^{\infty} |a_j|^{\frac{1}{2+\delta}} < \infty \quad (3.26)$$

for some $\delta > 0$. Then all statements in part (ii) remain valid.

Proof. (i) Consider the convergence of one-dimensional distributions:

$$\frac{1}{\sqrt{n^{3-\alpha}}} S_n \rightarrow_{law} \mathcal{N}(0, c_2). \quad (3.27)$$

In view of (2.14) and Lemma 1, relation (3.27) follows by Lindeberg's theorem provided

$$L_n := \sum_{s=-\infty}^n \mathbb{E}|Z_{s,n}|^{2+\delta} = o\left(n^{\frac{(3-\alpha)(2+\delta)}{2}}\right). \quad (3.28)$$

By Minkowski's inequality and assumptions (2.10) and (3.23) we obtain

$$\begin{aligned} \mathbb{E}|Z_{s,n}|^{2+\delta} &\leq \left(\sum_{k=1 \vee s}^n (\mathbb{E}|\gamma(a_{k-s})|^{2+\delta})^{\frac{1}{2+\delta}} \right)^{2+\delta} \\ &\leq C \left(\sum_{k=1 \vee s}^n |a_{k-s}|^{\frac{1}{2}} \right)^{2+\delta} \leq C \left(\sum_{k=1 \vee s}^n |k-s|_+^{-\frac{\alpha}{2}} \right)^{2+\delta} \end{aligned} \quad (3.29)$$

and therefore $L_n \leq C(L_n^- + L_n^+)$, where

$$\begin{aligned} L_n^- &= \sum_{s=-\infty}^0 \left(\sum_{k=1}^n |k-s|_+^{-\frac{\alpha}{2}} \right)^{2+\delta} = \sum_{s=0}^{\infty} \left(\sum_{k=1}^n (k+s)^{-\frac{\alpha}{2}} \right)^{2+\delta}, \\ L_n^+ &= \sum_{s=1}^n \left(\sum_{k=1}^n k^{-\frac{\alpha}{2}} \right)^{2+\delta} = n \left(\sum_{k=1}^n k^{-\frac{\alpha}{2}} \right)^{2+\delta}. \end{aligned}$$

Here, $L_n^+ = \mathcal{O}\left(n(n^{1-\frac{\alpha}{2}})^{2+\delta}\right) = o\left(n^{\frac{(3-\alpha)(2+\delta)}{2}}\right)$. The same relation for L_n^- follows from

$$\begin{aligned} L_n^- &\leq \int_0^\infty dx \left(\int_0^n (x+y)^{-\frac{\alpha}{2}} dx \right)^{2+\delta} = cn \left(n^{1-\frac{\alpha}{2}} \right)^{2+\delta}, \quad \text{with} \\ c &= \int_0^\infty dx \left(\int_0^1 (x+y)^{-\frac{\alpha}{2}} dx \right)^{2+\delta} < \infty. \end{aligned}$$

This proves (3.28) and the one-dimensional convergence in (3.27). Finite-dimensional convergence in (3.24) follows similarly using Cramér-Wold device. Finally, the tightness in $\mathcal{D}(J_1)$ of the partial sums process in (3.24) follows by Kolmogorov's criterion and from property (2.14) (see, e.g. [12], proposition 4.2.2). This proves part (i).

(ii) Again, it suffices to prove the convergence of one-dimensional distributions:

$$n^{-1/2} S_n \rightarrow_{law} \mathcal{N}(0, \sigma^2). \quad (3.30)$$

By writing S_n as in (3.22) and using Lindeberg's theorem relation (3.30) follows from

$$L_n = \sum_{s=-\infty}^n \mathbb{E}|Z_{s,n}|^{2+\delta} = o\left(n^{\frac{2+\delta}{2}}\right). \quad (3.31)$$

Using Minkowski's inequality and assumptions (3.23) and (2.16) similarly as in part (i) we obtain

$$\mathbb{E}|Z_{s,n}|^{2+\delta} \leq C \left(\sum_{k=1 \vee s}^n |a_{k-s}|^{\frac{1}{2}} \right)^{2+\delta} \quad (3.32)$$

$$\begin{aligned} &\leq C \left(\sum_{k=1 \vee s}^n |(k-s)a_{k-s}| \right)^{\frac{2+\delta}{2}} \left(\sum_{k=1 \vee s}^n (k-s)^{-1} \right)^{\frac{2+\delta}{2}} \\ &\leq C \left(\sum_{k=1 \vee s}^n (k-s)^{-1} \right)^{\frac{2+\delta}{2}}. \end{aligned} \quad (3.33)$$

and hence

$$\begin{aligned} \sum_{s=-n}^n \mathbb{E}|Z_{s,n}|^{2+\delta} &\leq Cn(\log n)^{\frac{2+\delta}{2}} = o(n^{\frac{2+\delta}{2}}), \\ \sum_{s=-\infty}^{-n} \mathbb{E}|Z_{s,n}|^{2+\delta} &\leq C \sum_{s=n}^{\infty} \left(\sum_{k=1}^n \frac{1}{k+s} \right)^{\frac{2+\delta}{2}} \leq C \sum_{s=n}^{\infty} (ns^{-1})^{\frac{2+\delta}{2}} \leq Cn = o(n^{\frac{2+\delta}{2}}), \end{aligned}$$

proving (3.31) and (3.30). To show the last statement of (ii) (the tightness in $D[0, 1]$), it suffices to prove the bound

$$\mathbb{E}|S_n|^{2+\delta} \leq Cn^{\frac{2+\delta}{2}}, \quad (3.34)$$

see ([12], proposition 4.4.4). By Rosenthal's inequality,

$$\mathbb{E}|S_n|^{2+\delta} \leq C \left(\sum_{s=-\infty}^n (\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}} \right)^{\frac{2+\delta}{2}}.$$

Using (3.32) and $\sum_{k=1}^{\infty} |a_k|^{\frac{1}{2}} < \infty$, we get $\max_{|s| \leq n} \mathbb{E}|Z_{s,n}|^{2+\delta} < C$ and

$$\begin{aligned} \sum_{s=-\infty}^{-n} (\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}} &\leq C \sum_{s=n}^{\infty} \left(\sum_{k=1}^n |a_{k+s}|^{\frac{1}{2}} \right)^2 \\ &\leq C \sum_{k_1, k_2=1}^n \sum_{s=n}^{\infty} |a_{k_1+s}|^{\frac{1}{2}} |a_{k_2+s}|^{\frac{1}{2}} \leq Cn. \end{aligned} \quad (3.35)$$

This proves (3.34) and part (ii), too.

(iii) Similarly as in (3.29) and using (3.26) we get

$$\mathbb{E}|Z_{s,n}|^{2+\delta} \leq C \left(\sum_{k=1 \vee s}^n |a_{k-s}|^{\frac{1}{2+\delta}} \right)^{2+\delta} \leq C \sum_{k=1 \vee s}^n |a_{k-s}|^{\frac{1}{2+\delta}} \leq C$$

for any $-\infty < s \leq n$ and hence

$$\begin{aligned} \sum_{s=-\infty}^{-n} \mathbb{E}|Z_{s,n}|^{2+\delta} &\leq C \sum_{s=n}^{\infty} \sum_{k=1}^n |a_{k+s}|^{\frac{1}{2+\delta}} \leq Cn, \\ \sum_{s=-\infty}^{-n} (\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}} &\leq C \sum_{s=n}^{\infty} \left(\sum_{k=1}^n |a_{k+s}|^{\frac{1}{2+\delta}} \right)^2 \leq Cn \end{aligned}$$

as in (3.35). Hence, (3.31) and (3.34) follow, proving part (iii) and completing the proof of Theorem 1. \square

Remark 2. The crucial condition for Gaussian partial sums limit under long-range dependence assumption (2.12) in Theorem 1 (i) is (3.23). Clearly this condition is satisfied for the Brownian motion $\gamma(u) = B(u)$, in which case $|\mu|_{2+\delta}(u) = \mathbb{E}|B(u)|^{2+\delta} = |u|^{\frac{2+\delta}{2}} \mathbb{E}|B(1)|^{2+\delta}$. On the other hand, condition (3.23) is not satisfied for most jump processes. Particularly, if $\gamma(u) = P(u) - u, u \geq 0$ is a centered Poisson process with intensity $\mathbb{E}P(u) = u$, then

$$|\mu|_{2+\delta}(u) = ue^{-u}|1 - u|^{2+\delta} + \mathcal{O}(u^{2+\delta} + u^2) \sim u \quad (u \rightarrow 0)$$

and (3.23) fails, but the first condition in (3.26) is satisfied. In particular, in the case of Poisson seed process, the trawl process satisfies Donsker's theorem if the trawl decays fast enough so that (3.26) holds.

Let us present further examples of seed processes satisfying the conditions in Theorem 1.

Example 4 (Geometric centered Brownian motion). Set $\gamma(u) = e^{B(u)-u/2} - 1, u \geq 0$, where B is a standard Brownian motion as above. We have $\mathbb{E}\gamma(u) = 0$ and (if $u \leq v$)

$$\begin{aligned} \rho(u, v) &= \mathbb{E} \exp\{B(u) + B(v) - \frac{u+v}{2}\} - 1 \\ &= \exp\left\{\left(\frac{1}{2}\mathbb{E}(B(u) + B(v))^2 - \frac{u+v}{2}\right)\right\} - 1 \\ &= \exp\left\{\left(\frac{1}{2}(u+v+2u) - \frac{u+v}{2}\right)\right\} - 1 \\ &= e^u - 1 = u \wedge v + \mathcal{O}((u \wedge v)^2), \quad u \wedge v \rightarrow 0. \end{aligned}$$

Therefore conditions (2.9), (2.11) are satisfied. We also have by Taylor's expansion that $|\mu|_4(u) = \mathbb{E}|e^{B(u)-u/2} - 1|^4 = e^{6u} - 4e^{3u} + 6e^u - 3 = \mathcal{O}(u^2), u \rightarrow 0$ so that (3.23) is satisfied with $\delta = 2$.

Example 5 (Diffusion process).

$$\gamma(u) = \int_0^u b(v)dB(v)$$

with B a Brownian motion, and $(b(v))_{v \geq 0}$ a random predictable process with $\lim_{v \rightarrow 0} \mathbb{E}b^2(v) = C > 0$. Then $g(u) = \int_0^u \mathbb{E}b^2(v)dv \sim Cu (u \rightarrow 0)$ and $\rho(u, v) = g(u), 0 \leq u \leq v$ so that conditions (2.9) and (2.11) are satisfied. Moreover, if $\mathbb{E}|b(v)|^{2+\delta} \leq C$ then by the moment inequality for Brownian integrals (see, e.g. [18], theorem 9.9.2)

$$\begin{aligned} |\mu|_{2+\delta}(u) &\leq C \mathbb{E} \left(\int_0^u b^2(v)dv \right)^{\frac{2+\delta}{2}} \\ &\leq C \left(\int_0^u \mathbb{E}|b(v)|^{2+\delta}dv \right) \left(\int_0^u 1dv \right)^{\frac{2+\delta}{2}-1} \leq Cu^{\frac{2+\delta}{2}}, \end{aligned}$$

hence assumption (3.23) holds, too.

3.2 Stable scenario (Case 2)

We assume now that seed process $\gamma = \{\gamma(u), u \geq 0\}$ is a piecewise constant nondecreasing process starting at $\gamma(0) = 0$ with unit jumps at points $0 = \tau_0 < \tau_1 < \tau_2 < \dots$:

$$\gamma(u) = \sum_{k=0}^{\infty} k \cdot \mathbf{1}(\tau_k \leq u < \tau_{k+1}) \quad (3.36)$$

and such that the distribution of the first jump-point $\tau_1 > 0$ has a bounded probability density $\theta(u)$:

$$\mathbb{P}(0 < \tau_1 \leq u) = \int_0^u \theta(y) dy \quad \text{and} \quad \lim_{u \rightarrow 0} \theta(u) = 1. \quad (3.37)$$

Moreover, we shall assume that there exists $\delta > 2(\alpha - 1)$ such that

$$\mathbb{E}\gamma(u)^{2+\delta} < \infty, \quad \forall u > 0, \quad (3.38)$$

$$\mathbb{E}\gamma(u)^2 \mathbf{1}(\tau_2 \leq u) = \mathcal{O}(u^2), \quad u \rightarrow 0. \quad (3.39)$$

Remark 3. The second condition in (3.37) can be replaced by $\lim_{u \rightarrow 0} \theta(u) = C > 0$ without loss of generality. Conditions (3.37)-(3.40) are very general and there are satisfied by many jump processes γ as this was sketched in the introduction. As shown below, these conditions also imply the conditions on γ in Proposition 2.

Remark that $(\tau_1 \leq u) = (\gamma(u) \geq 1)$, $(\tau_2 \leq u) = (\gamma(u) \geq 2)$ and therefore an alternative way to set condition (3.39) is $\mathbb{E}\gamma^2(u) \mathbf{1}(\gamma(u) > 1) = \mathcal{O}(u^2)$, as $u \rightarrow 0$.

Proposition 3. *For the seed process γ in (3.36), conditions (3.37)-(3.39) imply the assumptions (2.3) and (2.9) of Proposition 2 (i). In addition, if*

$$\mathbb{E}\gamma(v) \mathbf{1}(\tau_1 \leq u, \tau_2 \leq v) = o(u), \quad 0 \leq u \leq v \rightarrow 0, \quad (3.40)$$

then (2.11) is satisfied.

Proof. From (3.36) we have

$$\mathbf{1}(\tau_1 \leq u) \leq \gamma(u) \leq \mathbf{1}(\tau_1 \leq u) + \gamma(u) \mathbf{1}(\tau_2 \leq u) \quad (3.41)$$

and hence

$$\mathbb{P}(\tau_1 \leq u) \leq \mu(u) \leq \mathbb{P}(\mathbf{1}(\tau_1 \leq u) + \mathbb{E}\gamma(u) \mathbf{1}(\tau_2 \leq u))$$

From (3.37), $\mathbb{P}(0 < \tau_1 \leq u) = u(1 + o(1))$ and from (3.39),

$$\mathbb{E}\gamma(u) \mathbf{1}(\tau_2 \leq u) \leq \mathbb{E}\gamma^2(u) \mathbf{1}(\tau_2 \leq u) = \mathcal{O}(u^2).$$

Therefore,

$$\mu(u) = u(1 + o(1)) + \mathcal{O}(u^2) = u(1 + o(1)) \quad (u \rightarrow 0). \quad (3.42)$$

Similarly, for the second moment $\mu_2(u) = \mathbb{E}\gamma^2(u)$ from (3.41), (3.37), (3.39) we obtain

$$\mathbb{P}(\tau_1 \leq u) \leq \mu_2(u) \leq \mathbb{P}(\mathbf{1}(\tau_1 \leq u) + 2\mathbb{E}\gamma(u)\mathbf{1}(\tau_2 \leq u) + \mathbb{E}\gamma^2(u)\mathbf{1}(\tau_2 \leq u)),$$

implying $\mu_2(u) = u(1 + o(1)) + \mathcal{O}(u^2) = u(1 + o(1))$ ($u \rightarrow 0$) and

$$g(u) = \mu_2(u) - \mu^2(u) = u(1 + o(1)) \quad (u \rightarrow 0). \quad (3.43)$$

Clearly, (3.42) and (3.43) imply (2.3) and (2.9). Consider assumption (2.11). Since

$$\rho(u, v) = \mathbb{E}\gamma(u)\gamma(v) - \mu(u)\mu(v) = \mathbb{E}\gamma(u)\gamma(v) - uv(1 + o(1)) = \mathbb{E}\gamma(u)\gamma(v) + o(u \wedge v),$$

as $0 < u \leq v \rightarrow 0$, condition (2.11) follows from

$$\mathbb{E}\gamma(u)\gamma(v) = u(1 + o(1)), \quad 0 < u \leq v \rightarrow 0. \quad (3.44)$$

From (3.41) for $0 < u \leq v$ we obtain

$$\begin{aligned} \mathbb{P}(\tau_1 \leq u) &\leq \mathbb{E}\gamma(u)\gamma(v) \\ &\leq \mathbb{P}(\tau_1 \leq u) + \mathbb{E}\gamma(u)\mathbf{1}(\tau_2 \leq u) + \mathbb{E}\gamma(v)\mathbf{1}(\tau_1 \leq u, \tau_2 \leq v) + \mathbb{E}\gamma(u)\gamma(v)\mathbf{1}(\tau_2 \leq u) \end{aligned}$$

where $\mathbb{E}\gamma(u)\gamma(v)\mathbf{1}(\tau_2 \leq u) \leq (\mathbb{E}\gamma^2(u)\mathbf{1}(\tau_2 \leq u))^{\frac{1}{2}}(\mathbb{E}\gamma^2(v))^{\frac{1}{2}} \leq Cu(\mathbb{E}\gamma^2(v))^{\frac{1}{2}}$ and $\mathbb{E}\gamma^2(v) = \mu_2(v) = \mathcal{O}(v)$, see (3.37), (3.39). Hence and from (3.40) we have that

$$\mathbb{E}\gamma(u)\mathbf{1}(\tau_2 \leq u) + \mathbb{E}\gamma(v)\mathbf{1}(\tau_1 \leq u, \tau_2 \leq v) + \mathbb{E}\gamma(u)\gamma(v)\mathbf{1}(\tau_2 \leq u) = o(u)$$

implying (3.44) and (2.11), too. \square

Theorem 2. Assume that $a_j \geq 0$ satisfy the regular decay condition in (2.12) with exponent $1 < \alpha < 2$ and that the seed process in (3.36) satisfies conditions (3.37)-(3.39). Then

$$n^{-\frac{1}{\alpha}}(S_{[nt]} - \mathbb{E}S_{[nt]}) \rightarrow_{f.d.d.} L_\alpha(t), \quad (3.45)$$

where $L_\alpha(t), t \geq 0$ is a homogeneous α -stable Lévy process with characteristic function

$$\mathbb{E}e^{izL_\alpha(t)} = \exp \left\{ -t|z|^\alpha \frac{c_0\Gamma(2-\alpha)}{1-\alpha} \left(\cos(\pi\frac{\alpha}{2}) - i \cdot \text{sgn}(z) \sin(\pi\frac{\alpha}{2}) \right) \right\}, \quad z \in \mathbb{R}. \quad (3.46)$$

Proof. Denote

$$Z = \sum_{j=0}^{\infty} \gamma(a_j), \quad Z^* = \sum_{j=0}^{\infty} \mathbf{1}(\gamma(a_j) \geq 1) = \#\{j \geq 0 : a_j \geq \tau_1\}, \quad Z^{**} = Z - Z^*. \quad (3.47)$$

Then $Z \geq Z^* \geq 0$ and the series for Z in (3.47) converges a.s. in view of (3.42) and has finite mean:

$$\mathbb{E}Z = \sum_{j=0}^{\infty} \mu(a_j) \leq C \sum_{j=0}^{\infty} a_j < \infty.$$

We shall prove that the tail d.f. of r.v. Z decays regularly with exponent $\alpha \in (1, 2)$:

$$\mathbb{P}(Z > y) = c_0 y^{-\alpha}(1 + o(1)), \quad \text{as } y \rightarrow \infty. \quad (3.48)$$

Relation (3.48) follows from (3.47) and

$$\mathbb{P}(Z^* > y) = c_0 y^{-\alpha}(1 + o(1)) \quad \text{and} \quad \mathbb{P}(Z^{**} > y) = o(y^{-\alpha}), \quad \text{as } y \rightarrow \infty, \quad (3.49)$$

Consider the first relation in (3.49). Since $\mathbb{P}(Z^* > k - 1) \geq \mathbb{P}(Z^* > y) \geq \mathbb{P}(Z^* > k)$ when $k - 1 \leq y \leq k$, it suffices to show (3.49) for $y = k - 1$, or the probability $\mathbb{P}(Z^* \geq k), k \in \mathbb{N}_+$. As noted in the proof of Proposition 2, for any $\epsilon > 0$ there exists $j_0 > 0$ such that $c_0(1 - \epsilon)j^{-\alpha} < a_j < c_0(1 + \epsilon)j^{-\alpha}, \forall j \geq j_0$. Clearly, for any $k \geq 1$ we have $\mathbb{P}(Z_- \geq k + j_0) \leq \mathbb{P}(Z^* \geq k) \leq \mathbb{P}(Z_+ \geq k - j_0)$, where

$$\begin{aligned} Z_+ &= \sum_{j=j_0}^{\infty} \mathbf{1}(\tau_1 \leq c_0(1 + \epsilon)j^{-\alpha}) = \#\{j \geq j_0 : \tau_1 \leq c_0(1 + \epsilon)j^{-\alpha}\}, \\ Z_- &= \sum_{j=j_0}^{\infty} \mathbf{1}(\tau_1 \leq c_0(1 - \epsilon)j^{-\alpha}) = \#\{j \geq j_0 : \tau_1 \leq c_0(1 - \epsilon)j^{-\alpha}\}. \end{aligned}$$

According to (3.37), as $k \rightarrow \infty$,

$$\mathbb{P}(Z_+ \geq k - j_0) = \mathbb{P}(\tau_1 < c_0(1 + \epsilon)k^{-\alpha}) = \int_0^{c_0(1 + \epsilon)k^{-\alpha}} \theta(y)dy \sim c_0(1 + \epsilon)k^{-\alpha}$$

and, similarly,

$$\mathbb{P}(Z_- \geq k + j_0) = \mathbb{P}(\tau_1 < c_0(1 - \epsilon)(k + 2j_0 - 1)^{-\alpha}) \sim c_0(1 - \epsilon)k^{-\alpha}.$$

Therefore, $c_0(1 - \epsilon) \leq \liminf k^\alpha \mathbb{P}(Z^* \geq k) \leq \limsup k^\alpha \mathbb{P}(Z^* \geq k) \leq c_0(1 + \epsilon)$, where $\epsilon > 0$ is arbitrary small, proving the first relation in (3.49). To prove the second relation in (3.49), note $Z^{**} \leq \sum_{j=0}^{\infty} \gamma(a_j) \mathbf{1}(a_j \geq \tau_2)$ and then by (3.39) and Minkowski's inequality we obtain

$$\mathbb{E}^{\frac{1}{2}}(Z^{**})^2 \leq \sum_{j=0}^{\infty} (\mathbb{E} \gamma^2(a_j) \mathbf{1}(a_j \geq \tau_2))^{\frac{1}{2}} \leq C \sum_{j=0}^{\infty} |a_j| < \infty$$

proving the second relation in (3.49) and hence (3.48) as well. In turn, (3.48) implies that the distribution of r.v. Z belongs to the domain of attraction of asymmetric α -stable law, viz.,

$$n^{-\frac{1}{\alpha}} \sum_{k=1}^{[nt]} (Z_k - \mathbb{E} Z_k) \rightarrow_{f.d.d.} L_\alpha(t), \quad (3.50)$$

where $Z_k = \sum_{j=0}^{\infty} \gamma_k(a_j)$, $k \in \mathbb{Z}$ are i.i.d. copies of r.v. Z in (3.47) and L_α is the α -stable Lévy process in (3.24)-(3.46). See e.g. ([13], theorem 2.6.7).

Relation (3.45) follows from (3.50) if we show that the partial sums process in (3.45) can be approximated by the partial sums process in (3.50), in the sense that

$$\mathbb{E}|S_n - \tilde{S}_n| = o(n^{\frac{1}{\alpha}}), \quad \text{where} \quad \tilde{S}_n = \sum_{k=1}^n Z_k. \quad (3.51)$$

We have $\tilde{S}_n - S_n = R'_n - R''_n$, where

$$R'_n = \sum_{1 \leq s \leq n} \sum_{j > n-s} \gamma_s(a_j), \quad R''_n = \sum_{s \leq 0} \sum_{1 \leq k \leq n} \gamma_s(a_{k-s}),$$

then $R'_n \geq 0, R''_n \geq 0$.

Using (3.42) and (2.12) we obtain

$$\begin{aligned} \mathbb{E}R'_n &= \sum_{1 \leq s \leq n} \sum_{j > n-s} \mathbb{E}\gamma_s(a_j) = \sum_{1 \leq s \leq n} \sum_{j > n-s} \mu(a_j) \\ &\leq C \sum_{1 \leq s \leq n} \sum_{j > n-s} j^{-\alpha} = \mathcal{O}(n^{2-\alpha}), \\ \mathbb{E}R''_n &= \sum_{s \leq 0} \sum_{1 \leq k \leq n} \mathbb{E}\gamma_s(a_{k-s}) = \sum_{s \leq 0} \sum_{1 \leq k \leq n} \mu(a_{k-s}) \\ &= \sum_{s \geq 0} \sum_{1 \leq k \leq n} (k+s)^{-\alpha} = \mathcal{O}(n^{2-\alpha}), \end{aligned}$$

implying (3.51) since $2 - \alpha < 1/\alpha$ for $1 < \alpha < 2$. Theorem 2 is proved. \square

Example 6 (Jump processes and the assumptions of Theorem 2). For such a jump process $(\gamma(u) = k) = (\tau_k \leq u < \tau_{k+1})$. Conditions in (3.37)-(3.39) on the seed process $\{\gamma(u), u \geq 0\}$ in Theorem 2 are rather weak and essentially involve the distribution of the first jump-point τ_1 provided the second jump τ_2 cannot occur very fast after τ_1 . Particularly,

- The Bernoulli process is very simple: in this case $\tau_2 = \infty$ thus $\mathbb{P}(\tau_2 \leq u) = \mathbb{P}(\tau_2 \leq v) = 0$,
- The Poisson process in Example 3. Indeed, for $\gamma(u) = P(u)$ (3.39) holds since

$$\mathbb{E}\gamma(u)^2 \mathbb{1}(\tau_2 \leq u) = \mathbb{E}\gamma(u)^2 - \mathbb{P}(\gamma(u) = 1) = u + u^2 - ue^{-u} = \mathcal{O}(u^2).$$

Verification of (3.40) for $\gamma(u) = P(u)$ is slightly more involved, as follows. Let $b(u, v) = \mathbb{E}\gamma(v) \mathbb{1}(\tau_1 \leq u, \tau_2 \leq v) = b_1(u, v) + b_2(u, v)$, where $b_1(u, v) = \mathbb{E}\gamma(v) \mathbb{1}(\tau_2 \leq u) \leq \mathbb{P}^{2/3}(\tau_2 \leq u) \mathbb{E}^{1/3}\gamma^3(v) = \mathcal{O}(u^{4/3}) = o(u)$ since $\mathbb{P}(\tau_2 \leq u) = \mathbb{P}(\gamma(u) \geq 2) = \mathcal{O}(u^2), u \rightarrow 0$. Next, since $\tau_2 > u$ implies $\gamma(u) = 1$ so $b_2(u, v) = \mathbb{E}\gamma(v) \mathbb{1}(\tau_1 \leq u, u < \tau_2 \leq v) = \mathbb{P}(\tau_1 \leq u, u < \tau_2 \leq v) + \mathbb{E}(\gamma(v) - \gamma(u)) \mathbb{1}(\gamma(u) = 1, \gamma(v) \geq 1)$, where $\mathbb{P}(\tau_1 \leq u, u < \tau_2 \leq v) = \mathbb{P}(\gamma(u) = 1, \gamma(v) - \gamma(u) \geq 1) = \mathcal{O}(u(v-u)) = o(u)$ and, similarly $\mathbb{E}(\gamma(v) - \gamma(u)) \mathbb{1}(\gamma(u) = 1, \gamma(v) \geq 1) = \mathbb{P}(\gamma(u) = 1) \mathbb{E}\gamma(v-u) = \mathcal{O}(u(v-u)) = o(u), 0 < u \leq v \rightarrow 0$, proving (3.40).

- Other examples of jump processes satisfying (3.37)-(3.39) include mixed Poisson processes (Example 3) and renewal process with independent intervals τ_1 and $\tau_2 - \tau_1$ and $\mathbb{P}(\tau_2 - \tau_1 \leq x) = \mathcal{O}(x)$ since

$$\mathbb{P}(\tau_2 \leq u) = \int_0^u \theta(y) \mathbb{P}(\tau_2 - \tau_1 \leq u - y) dy \leq C \int_0^u (u - y) dy = \mathcal{O}(u^2)$$

as in the Poisson case. The same conditions also holds for mixed Poisson processes driven by some random variable $\zeta > 0$ (Example 3). (thus again the case of negative binomials fits our result as sketched in [6]).

We note that the functional convergence in (3.45) is open and may not hold in the J_1 -topology. At the cost of additional structure we can prove the convergence in Skorohod's M_1 -topology. For definitions and properties related to association of random variables we refer to [10].

Theorem 3. *Suppose that all assumptions of Theorem 2 hold. If the jump random variables τ_1, τ_2, \dots are associated (in particular, if they are sums of independent positive random variables) then the finite-dimensional convergence (3.45) can be strengthened to*

$$n^{-\frac{1}{\alpha}}(S_{[nt]} - \mathbb{E}S_{[nt]}) \rightarrow_{\mathcal{D}(M_1)} L_\alpha(t), \quad (3.52)$$

Proof. By ([22], theorem 1) it suffices to verify that X_1, X_2, X_3, \dots are associated random variables. By ([10], property P_5) it is enough to check association of

$$\sum_{j=0}^N \gamma_{1-j}(a_j), \quad \sum_{j=0}^N \gamma_{2-j}(a_j), \dots, \quad \sum_{j=0}^N \gamma_{k-j}(a_j),$$

for each $N \in \mathbb{N}$ and $k \in \mathbb{N}$, where $\gamma_j(\cdot)$ are independent copies of (3.36). This in turn is implied by ([10], properties P_4 and P_2), provided the family $\gamma(a_1), \gamma(a_2), \gamma(a_3), \dots$, is associated. But

$$\gamma(u) = \sum_{k=0}^{\infty} k \cdot \mathbb{1}(\tau_k \leq u < \tau_{k+1}) = \sum_{k=1}^{\infty} \mathbb{1}(\tau_k \leq u),$$

and by arguments already presented above it is enough to prove association of random variables

$$\begin{array}{cccc} \mathbb{1}(\tau_1 \leq a_0), & \mathbb{1}(\tau_2 \leq a_0), & \cdots & \mathbb{1}(\tau_k \leq a_0), \\ \mathbb{1}(\tau_1 \leq a_1), & \mathbb{1}(\tau_2 \leq a_1), & \cdots & \mathbb{1}(\tau_k \leq a_1), \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}(\tau_1 \leq a_N), & \mathbb{1}(\tau_2 \leq a_N), & \cdots & \mathbb{1}(\tau_k \leq a_N), \end{array} \quad (3.53)$$

for each $N \in \mathbb{N}$ and $k \in \mathbb{N}$. Let us notice that $\mathbb{1}(\tau_j \leq a_m) = 1 - \mathbb{1}(\tau_j > a_m)$ and that functions $f_m(x) = \mathbb{1}(x > a_m)$ are nondecreasing. Therefore, if τ_1, τ_2, \dots are associated, then also the family $\{f_m(\tau_j); j, m \in \mathbb{N}\}$ is associated. By ([10], property BP_1) array (3.53) is associated as well. \square

Remark 4. Let us consider two families $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$ of processes of the form (3.36). Consider stationary processes X_1^+, X_2^+, \dots , and X_1^-, X_2^-, \dots , each built according to the recipe (2.5), and the corresponding partial sum processes $S_{[nt]}^+$ and $S_{[nt]}^-$.

If $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$ are independent and

$$n^{-\frac{1}{\alpha}}(S_{[nt]}^+ - \mathbb{E}S_{[nt]}^+) \rightarrow_{f.d.d.} L_\alpha^+(t), \quad n^{-\frac{1}{\alpha}}(S_{[nt]}^- - \mathbb{E}S_{[nt]}^-) \rightarrow_{f.d.d.} L_\alpha^-(t), \quad (3.54)$$

then also

$$n^{-\frac{1}{\alpha}} \left((S_{[nt]}^+ - S_{[nt]}^-) - \mathbb{E}(S_{[nt]}^+ - S_{[nt]}^-) \right) \rightarrow_{f.d.d.} L_{\alpha}(t), \quad (3.55)$$

where $L_{\alpha} \sim L^+ - L^-$ with independent $L^+ \sim L_{\alpha}^+$ and $L^- \sim L_{\alpha}^-$.

In particular, if γ_j^+ and γ_j^- are identically distributed, then the resulting trawl process is centered and the limiting Lévy process is *symmetric*. This is the case if e.g. γ^{\pm} are both homogeneous Poisson processes with identical intensities or Bernoulli processes $\gamma^{\pm}(u) = \mathbb{1}(U^{\pm} \leq u)$ for independent uniform rvs, U^{\pm} .

Remark 5. As the example of an ordinary moving average with summable coefficients shows, (3.54) may imply (3.55) without the assumption of independence of $S_{[nt]}^+$ and $S_{[nt]}^-$ (see e.g. ([1], corollary 2.2)). In the functional limit theorem given below we follow this general approach and obtain the functional convergence in the non-Skorohodan S topology (see [14]). We shall denote by $\rightarrow_{\mathcal{D}(S)}$ the weak convergence in the Skorohod space $D[0, 1]$ equipped with the S topology).

Corollary 1. *In the framework of Remark 4, suppose that both $S_{[nt]}^+$ and $S_{[nt]}^-$ satisfy all assumptions of Theorem 3, so that*

$$n^{-\frac{1}{\alpha}}(S_{[nt]}^+ - \mathbb{E}S_{[nt]}^+) \rightarrow_{\mathcal{D}(M_1)} L_{\alpha}^+(t), \quad n^{-\frac{1}{\alpha}}(S_{[nt]}^- - \mathbb{E}S_{[nt]}^-) \rightarrow_{\mathcal{D}(M_1)} L_{\alpha}^-(t), \quad (3.56)$$

for some α -stable Lévy motions L_{α}^+ and L_{α}^- .

If for some càdlàg stochastic process K we have

$$n^{-\frac{1}{\alpha}} \left((S_{[nt]}^+ - S_{[nt]}^-) - \mathbb{E}(S_{[nt]}^+ - S_{[nt]}^-) \right) \rightarrow_{f.d.d.} K(t), \quad (3.57)$$

then

$$n^{-\frac{1}{\alpha}} \left((S_{[nt]}^+ - S_{[nt]}^-) - \mathbb{E}(S_{[nt]}^+ - S_{[nt]}^-) \right) \rightarrow_{\mathcal{D}(S)} K(t).$$

Proof. By ([1], theorem 3.13) (3.56) implies the uniform S -tightness of the corresponding processes. The proof of ([1], proposition 3.16) gives the uniform S -tightness of the differences. A direct application of ([1], proposition 3.3) concludes the proof. \square .

Acknowledgements. This study begun with a question from Wilfredo Palma (Santiago de Chile) to the first author: *how to define LRD integer valued models?* We wish to thank him for considering this problem.

This work has been developed within the MME-DII center of excellence (ANR-11-LABEX-0023-01) and was partially supported by CNPq-Brazil.

We also thank the Universities UFRGS (Porto Alegre) and Nicolaus Copernicus (Toruń) for their support.

References

- [1] Balan, R., Jakubowski, A. and Louhichi, S. (2016) Functional convergence of linear processes with heavy-tailed innovations. *J. Theoret. Probab.* 29, 491–526.
- [2] Barndorff-Nielsen, O. E. (2010) Stationary infinitely divisible processes. *REBRAPE Braz. J. Probab. Stat.* 25, 294–322.
- [3] Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. (2011) Recent advances in ambit stochastics. Preprint available at arXiv:1210.1354.
- [4] Barndorff-Nielsen, O.E., Lunde, A., Shepard, N. and Veraart, A.E.D. (2014) Integer-valued trawl processes: a class of stationary infinitely divisible processes. *Scand. J. Statist.* 41, 693–724.
- [5] Billingsley, P.. (1999) *Convergence of Probability Measures*. 2nd ed., Wiley, New York.
- [6] Christou, V. and Fokianos, K. (2014) Quasi-likelihood inference for negative binomial time series. *J. Time Series Anal.* 35, 55–78.
- [7] Davydov, Y. A. (1970) The invariance principle for stationary processes. *Theor. Probab. Appl.* 15, 487–498.
- [8] Dehling, H. and Philipp, W. (2002) Empirical process techniques for dependent data. In: H. Dehling, T. Mikosch and M. Sørensen (Eds.), *Empirical Process Techniques for Dependent Data*, pp. 1–113. Birkhäuser, Boston.
- [9] Doukhan, P. , Oppenheim, G. and Taqqu M. S. (Eds.)(2003) *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston.
- [10] Esary, J.D., Proschan, F. and Walkup, D.W. (1967) Association of random variables, with applications. *Ann. Math. Statist.* 38, 1466–1474.
- [11] Feller, W. (1966) *An Introduction to Probability Theory and Its Applications*, vol. 2. Wiley, New York.
- [12] Giraitis, L., Koul, H. L. and Surgailis, D. (2012) *Large Sample Inference for Long Memory Processes*. Imperial College Press, London.
- [13] Ibragimov, I.A. and Linnik, Y.V. (1971) *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.
- [14] Jakubowski, A. (1997) A non-Skorohod topology on the Skorohod space. *Electron. J. Probab.* 2, 1–21.
- [15] Hall, P., Koul, H.L. and Turlach, B.A. (1997) Note on convergence rates of semiparametric estimators of dependence index. *Ann. Statist.* 25, 1725–1739.
- [16] Kaj, I. and Taqqu, M. S. (2008) Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In: Vares, M.E. and Sidoravicius, V. (Eds.) *An Out of Equilibrium 2*. Progress in Probability, vol. 60, pp. 383–427. Birkhäuser, Basel.
- [17] Konstantopoulos, T. and Lin, S.-J. (1998) Macroscopic models for long-range dependent network traffic. *Queueing Systems* 28, 215–243.
- [18] Kwapien, S. and Woyczyński, W. A. (1992) *Random Series and Stochastic Integrals: Single and Multiple*. Birkhäuser, Boston.
- [19] Leipus, R. and Surgailis, D. (2003) Random coefficient autoregression, regime switching and long memory. *Adv. Appl. Probab.* 35, 737–754.
- [20] Leipus, R., Paulauskas, V. and Surgailis, D. (2005) Renewal regime switching and stable limit laws. *J. Econometrics* 129, 299–327.
- [21] Lifshits, M. (2014) *Random Processes by Example*. World Scientific, New Jersey.

- [22] Louhichi, S. and Rio, E. (2011) Functional convergence to stable Lévy motions for iterated random Lipschitz mappings. *Electron. J. Probab.* 16, 2452–2480.
- [23] Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* 12, 23–68.
- [24] Pilipauskaitė, V. and Surgailis, D. (2014) Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.* 124, 1011–1035.
- [25] Resnick, S. and Van den Berg, E. (2000) Weak convergence of high-speed traffic models. *J. Appl. Prob.* 37, 375–397.
- [26] Skorohod, A.V. (1956) Limit theorems for stochastic processes. *Theory Probab. Appl.* 1, 261–290.
- [27] Surgailis, D. (2004) Stable limits of sums of bounded functions of long memory moving averages with finite variance. *Bernoulli* 10, 327–355.
- [28] Taqqu, M.S. and Levy, J.B. (1986) Using renewal processes to generate long-range dependence and high variability. In: Eberlein, E. and Taqqu, M.S. (Eds.) *Dependence in Probability and Statistics*, pp. 51–72. Birkhäuser, Boston.
- [29] Taqqu, M.S., Willinger, W. and Sherman, R. (1997) Proof of the fundamental result in self-similar traffic modeling. *Computer Commun. Rev.* 27, 5–23.
- [30] Willinger, W., Paxon, V., Riedi, R.H. and Taqqu, M.S. (2003) Long-range dependence and data network traffic. In: Doukhan, P., Oppenheim, G. and Taqqu, M.S. (Eds.) *Theory and Applications of Long-Range Dependence*, pp. 373–407. Birkhäuser, Boston.
- [31] Wolpert, R. L. and Taqqu, M. S. (2005) Fractional Ornstein-Uhlenbeck Lévy processes and the Telecom process: upstairs and downstairs. *Signal Process.* 85, 1523–1545.